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Paper No. : Electromagnetic Theory
Module : Electromagnetic Waves - II

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Description of Module	
Subject Name	Physics
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Contents

Learning Objectives:

In this module we continue with study of propagation of electromagnetic waves, and students may get to know about the following:

1. The reflection and refractiuon of the electromagnetic waves at the interface between two media.
2. The kinematic properties like geometrical relations between sngles of incidence, reflection and refraction.
3. The dynamic relations like the intensity of the reflected and refracted components of the wave.
4. Polarization of an unpolarized incident wave after reflection and refraction.
5. Conditions for total internal reflection of an incident wave.

Electromagnetic Waves - II

8.1 Reflection and Refraction of Electromagnetic Waves

In this module we study the reflection and refraction of electromagnetic waves at a plane interface between two dielectric materials, the most familiar phenomena in optics. In the history of physics there has been a big debate over the nature of light which lasted for almost a century and a half. The phenomena of reflection and refraction played a big role in this debate. Newton developed and championed his corpuscular hypothesis, arguing that the perfectly straight lines of reflection demonstrated light's particle nature. On the other hand, Newton's contemporaries Robert Hooke and Huygens—and later Fresnel—mathematically refined the wave viewpoint, showing that if light traveled at different speeds in different media (such as water and air), refraction could be easily explained as the medium-dependent propagation of light waves. The discovery of double-slit interference, was the beginning of the end for the particle light camp. And then came Maxwell's equations which had a wave-like solution with a speed equal to the speed of light. And finally the explanation of both the kinematic and dynamic aspects of the phenomena of reflection and refraction from the first principles marked the end of any debate about the wave nature of light.

By kinematic properties of reflection and refraction we mean the relations between angles of incidence, reflection and refraction etc., and by dynamic properties we mean the intensities of reflected and refracted waves and phenomenon of polarization etc.

8.1.1 The Kinematic Properties

The kinematic properties follow immediately from the wave nature of the phenomenon and the fact that there are certain boundary conditions to be satisfied but do not depend on the specific nature of these boundary conditions. The dynamic properties, on the other hand, depend entirely on the specific nature of these boundary conditions. [See figure 7.5 from Jackson Ed. 2]

Let the plane z=0 be the interface between two dielectric media. The medium below has permeability μ and permitivity ε, and that above μ' and ε' respectively. The quantity $\sqrt{\mu\varepsilon}$ is the refractive index of a medium. The velocity of light in a medium is $v = \frac{c}{\sqrt{\mu\varepsilon}}$. The refractive index of the medium below the interface (medium 1) is $n = \sqrt{\mu\varepsilon}$ and that of the medium above (medium 2) is $n' = \sqrt{\mu'\varepsilon'}$. Let a plane wave with angular frequency ω and wave vector $\vec{k}$ be incident from the medium 1 onto the interface. Let the refracted wave and the reflected wave have wave vectors $\vec{k}'$ and $\vec{k}''$ respectively and let $\hat{n}$ be the unit vector from medium 1 into 2. For reasons that we will make clear presently, the angular frequency of all the three waves, the incident, the refracted and the reflected, is the same. The magnitude of the wave numbers is related to the angular frequency by the relations

$$\begin{aligned} |\vec{k}| = |\vec{k}''| = k = \frac{\omega}{v} = \omega\sqrt{\mu\varepsilon} \\ |\vec{k}'| = k' = \frac{\omega}{v'} = \omega\sqrt{\mu'\varepsilon'} \end{aligned} \qquad (1)$$

The electric and the magnetic fields in an electromagnetic wave are related by

$$\vec{B} = \sqrt{\mu\varepsilon}\,\hat{n} \times \vec{E} = \sqrt{\mu\varepsilon}\,\frac{\vec{k} \times \vec{E}}{k} \qquad (2)$$

Hence the electric and the magnetic fields of the incident, refracted and reflected waves are represented by

Incident Wave:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{x}-\omega t)}$$

$$\vec{B} = \sqrt{\mu\varepsilon}\,\frac{\vec{k}\times\vec{E}}{k} = \vec{B}_0 e^{i(\vec{k}.\vec{x}-\omega t)}, \vec{B}_0 = \sqrt{\mu\varepsilon}\,\frac{\vec{k}\times\vec{E}_0}{k} = \frac{c}{\omega}\vec{k}\times\vec{E}_0 \qquad (3)$$

Refracted Wave:

$$\vec{E}' = \vec{E}_0' e^{i(\vec{k}'.\vec{x}-\omega t)}$$

$$\vec{B}' = \sqrt{\mu'\varepsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'} = \vec{B}_0' e^{i(\vec{k}'.\vec{x}-\omega t)}, \vec{B}_0' = \sqrt{\mu'\varepsilon'}\,\frac{\vec{k}'\times\vec{E}_0'}{k'} = \frac{c}{\omega}\vec{k}'\times\vec{E}_0' \qquad (4)$$

Reflected Wave:

$$\vec{E}'' = \vec{E}_0'' e^{i(\vec{k}''.\vec{x}-\omega t)}$$

$$\vec{B}'' = \sqrt{\mu\varepsilon}\,\frac{\vec{k}''\times\vec{E}''}{k} = \vec{B}_0'' e^{i(\vec{k}''.\vec{x}-\omega t)}, \vec{B}_0'' = \sqrt{\mu\varepsilon}\,\frac{\vec{k}''\times\vec{E}_0''}{k''} = \frac{c}{\omega}\vec{k}''\times\vec{E}_0'' \qquad (5)$$

Here v is the velocity of the wave propagation in medium 1 and v' in medium 2. The existence of the boundary conditions at z=0, and the fact that the boundary conditions must be satisfied at all points on the plane and at all times implies that the time and space variation of all fields must be the same at z=0. Thus, as we had anticipated, the frequency of both the refracted and reflected waves must remain unchanged. From the equality of the phases at z=0, we further have

$$(\vec{k}.\vec{x})_{z=0} = (\vec{k}'.\vec{x})_{z=0} = (\vec{k}''.\vec{x})_{z=0} \qquad (6)$$

independent of the nature of the boundary conditions. Equation (6) contains the kinematic aspects of refraction and reflection. First of all we see that all three wave vectors must lie in a plane. Furthermore, since $\hat{n}$ is the normal to the plane z=0, for any point $\vec{r}$ lying in the plane, $\hat{n}.\vec{r} = 0$. Hence

$$\hat{n}\times(\hat{n}\times\vec{r}) = (\hat{n}.\vec{r})\hat{n} - \vec{r} = -\vec{r}$$

on the interface. From equation (6)

$$(\vec{k}-\vec{k}'').\vec{r}=0 \Rightarrow (\vec{k}-\vec{k}'').[\hat{n}\times(\hat{n}\times\vec{r})]=0$$

or

$$(\vec{k}-\vec{k}'')\times\hat{n}.[(\hat{n}\times\vec{r})]=0 \tag{7a}$$

Similarly,

$$(\vec{k}-\vec{k}')\times\hat{n}.[(\hat{n}\times\vec{r})]=0 \tag{7b}$$

This is vector form of the kinematic properties of reflection and refraction. In more familiar terms of the angle of incidence i, angle of refraction r, and the angle of reflection r', this takes the form

$$k\sin i = k'\sin r = k''\sin r' \tag{8}$$

and since, $k = k''$, we get $i = r'$, i.e., the angle of incidence is equal to the angle of reflection. Furthermore we have

$$\frac{\sin i}{\sin r}=\frac{k'}{k}=\sqrt{\frac{\mu'\varepsilon'}{\mu\varepsilon}}=\frac{n'}{n}, \tag{9}$$

the well-known Snell's law.

8.1.2 The Dynamic Properties

For the dynamic properties one needs the specific boundary conditions to be applied. These boundary conditions are well known and they are valid for static as well as time-varying fields – normal components of $\vec{D}$ and $\vec{B}$ are continuous; tangential components of $\vec{E}$ and $\vec{H}$ are continuous across the interface. For the fields given by equations (2)-(4), these boundary conditions at z=0 lead to

$$[\vec{D}_0 + \vec{D}_0'' - \vec{D}_0].\hat{n} = 0 \Rightarrow [\varepsilon(\vec{E}_0 + \vec{E}_0'') - \varepsilon' \vec{E}_0'].\hat{n} = 0 \tag{10}$$

$$[\vec{B}_0 + \vec{B}_0'' - \vec{B}_0'].\hat{n} = 0 \Rightarrow [\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' - \vec{k}' \times \vec{E}_0'].\hat{n} = 0 \tag{11}$$

$$(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0') \times \hat{n} = 0 \tag{12}$$

$$(\vec{H}_0 + \vec{H}_0'' - \vec{H}_0') \times \hat{n} = 0 \Rightarrow [\frac{1}{\mu}(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'') - \frac{1}{\mu'}(\vec{k}' \times \vec{E}_0')] \times \hat{n} = 0 \tag{13}$$

In applying these boundary conditions it is convenient to consider two linear polarization state of the radiation separately. Then the situation for any arbitrary polarization can be obtained by a linear combination of the two. The vectors $\vec{k}$ and $\hat{n}$ define the plane of incidence (which is also the plane of reflection and refraction, as we have already seen). In one case the incident plane polarized plane wave has its plane of polarization (the plane that contains the electric field) perpendicular to plane of incidence. In the other case, the polarization vector is in the plane of incidence.

8.1.2.1 Polarization Perpendicular to the Plane of Incidence [See figure 7.6 from Jackson Ed. 2]

In this case all the electric fields are shown directed away from the viewer. The direction of the magnetic field is then determined from equation (2). Since the electric field is perpendicular to the plane of incidence, it is parallel to the surface. So $\vec{E}_0.\hat{n} = \vec{E}_0'\hat{n} = \vec{E}_0''\hat{n} = 0$, and equation (10) does not yield anything. Further, in this case the electric field is perpendicular to both $\vec{k}$ and $\hat{n}$, so that $\hat{n} \times \vec{k}$ is along $\vec{E}$, and

$$(\vec{k} \times \vec{E}_0).\hat{n} = \vec{E}_0.(\hat{n} \times \vec{k}) = E_0 k \sin i$$

Similarly

$$(\vec{k}'' \times \vec{E}_0'').\hat{n} = E_0'' k'' \sin i = E_0'' k \sin i;$$
$$(\vec{k}' \times \vec{E}_0').\hat{n} = E_0' k' \cos r = E_0' k \sin i$$

Hence equation (11) becomes

$$E_0 + E_0'' - E_0' = 0 \tag{14}$$

Since the electric field is normal to $\hat{n}$, equation (12) yields exactly the same result. The electric field being perpendicular to the normal, $(\vec{k} \times \vec{E}_0) \times \hat{n} = (\vec{k}.\hat{n}) E_0 = k E_0 \cos i$. Using this and the equivalent relations in the other two terms, equation (13) becomes

$$\frac{1}{\mu}(k E_0 \cos i - k'' E_0'' \cos i) - \frac{1}{\mu'} k'' E_0' \cos r = 0$$

and since $\frac{k'}{k} = \sqrt{\frac{\mu' \varepsilon'}{\mu \varepsilon}}$, $n = \sqrt{\mu \varepsilon}; n' = \sqrt{\mu' \varepsilon'}$ we have

$$\frac{n}{\mu}(E_0 - E_0'') \cos i - \frac{n'}{\mu'} E_0' \cos r = 0 \tag{15}$$

Equations (14) and (15) are the two relations that we get from the specific boundary conditions that the electric and magnetic fields are obliged to satisfy. These two equations can be solved for the relative magnitudes of the refracted and the reflected waves:

$$\frac{E_0'}{E_0} = \frac{\frac{2n}{\mu}\cos i}{\frac{n}{\mu}\cos i + \frac{n'}{\mu'}\cos r} = \frac{2n \cos i}{n \cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \tag{16}$$

$$\frac{E_0''}{E_0} = \frac{n \cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}}{n \cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \tag{17}$$

8.1.2.2 Polarization in the Plane of Incidence [See figure 7.7 from Jackson Ed. 2]

Now we consider the case when the electric field is in the plane of incidence. In this case $\vec{E}$, $\hat{n}$ and $\vec{k}$ are all three in the same plane. Let $\hat{t}$ be the unit vector in the plane of incidence but perpendicular to $\hat{n}$ and $\hat{u}$ be the unit vector perpendicular to the plane of incidence. Then

$$\vec{E}_0.\hat{n} = E_0 \sin i; \qquad \vec{E}_0 \times \hat{n} = \hat{u} E_0 \cos i;$$
$$(\vec{k} \times \vec{E}_0).\hat{n} = 0; \qquad (\vec{k} \times \vec{E}_0) \times \hat{n} = \hat{t} k E_0$$

Thus equation (10) reduces to

$$\varepsilon(E_0 + E_0'')\sin i - \varepsilon' E_0' \sin r = 0$$

Or on using Snell's law

$$\sqrt{\frac{\varepsilon}{\mu}}(E_0 + E_0'') - \sqrt{\frac{\varepsilon'}{\mu'}} E_0' = 0 \qquad (18)$$

Equation (11) yields nothing. From equation (12) we obtain

$$(E_0 - E_0'')\cos i - E_0' \cos r = 0 \qquad (19)$$

Equation (13) repeats the result of equation (10). The two equations (18) and (19) can be solved for the relative magnitudes of the refracted and reflected waves:

$$\frac{E_0'}{E_0} = \frac{2\cos i}{\cos r + \sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}\cos i} = \frac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \qquad (20)$$

$$\frac{E_0''}{E_0}=\frac{\cos i-\sqrt{\frac{\mu'\varepsilon}{\mu\varepsilon'}}\cos r}{\cos i+\sqrt{\frac{\mu'\varepsilon}{\mu\varepsilon'}}\cos r}=\frac{\frac{\mu}{\mu'}n'^2\cos i-n\sqrt{n'^2-n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i+n\sqrt{n'^2-n^2\sin^2 i}} \qquad (21)$$

The relative intensity relations (16 - 17) and (20 - 21) are known as Fresnel's formulas. For normal incidence (i=0), both these sets of equations reduce to

$$\frac{E_0'}{E_0}=\frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1}=\frac{2n}{n'+n} \qquad (22)$$

$$\frac{E_0''}{E_0}=\frac{1-\sqrt{\frac{\mu'\varepsilon}{\mu\varepsilon'}}}{1+\sqrt{\frac{\mu'\varepsilon}{\mu\varepsilon'}}}=\frac{n'-n}{n'+n} \qquad (23)$$

The last form in both the equations is obtained for the case $\mu=\mu'$. For dielectrics this condition is usually met. For the rest of this discussion, we will now assume $\mu=\mu'$.

8.2 Polarization by Reflection

Let us study the results obtained in some detail now. From equation (17) we see that for the case of polarization perpendicular to the plane of incidence, the reflected amplitude is never zero (unless, of course, $n=n'$; in which case there is no interface). However, for polarization parallel to the plane of incidence, there is an angle of incidence, called Brewster angle i_B, at which there is no reflected wave. From equation (21) this happens when

$$n'^2\cos i=n\sqrt{n'^2-n^2\sin^2 i}\Rightarrow i_B=\tan^{-1}(\frac{n'}{n})$$

From Snell's law, this is equivalent to $i_B+r_B=90^0$, i.e., the reflected and the refracted waves are exactly at right angles to each other. For reflection from lighter (denser) to a denser (lighter) medium this angle is greater (lesser) than 45^0. For a typical value of $(n'/n) = 1.5$, $i_B \approx 56^0$. If a plane wave of mixed polarization is incident on a plane interface at the Brewster angle, the reflected radiation will be completely plane polarized, with polarization vector perpendicular to the plane of incidence. This fact can be utilized to produce beams of pure plane-polarized light though it is not a very efficient way of doing so because at this angle most of the light polarized perpendicular to the plane of incidence is also transmitted so that the over-all intensity is rather poor. Even at angles other than the Brewster angle, particularly close to it, there is a tendency for the reflected wave to be predominantly polarized perpendicular to the plane of incidence. Thus the reflected beam is partially polarized even at other angles.

- Polarized sunglasses take advantage of this fact – the lenses in the glasses include a thin film polarizer that selectively absorbs light with perpendicular polarization. This greatly reduces glare from reflection off ice or water surface.
- Photographers use the same principle to remove reflections from water so that they can photograph objects beneath the surface. In this the polarizing filter camera can be rotated to be at the correct angle.

8.3 Total Internal Reflection

There is a whole range of angles for which there may not be any transmission of the wave and there is total internal reflection. This follows from kinematic properties, viz., Snell's law and applies equally to all polarizations. This phenomenon is called total internal reflection, and as the word "internal" implies, occurs when wave is incident from a medium with larger refractive index ($n > n'$). From Snell'd law, if $n > n'$, then r>i. Consequently, for some angle of incidence, the angle of refraction becomes $\pi/2$. This happens when

$$i = i_c = \sin^{-1}(\frac{n'}{n}) .$$

The angle i_c is called the critical angle of incidence. For waves incident at $i=i_c$, the refracted wave travels parallel to the surface. There can be no energy flow across the surface. Hence at that angle of incidence there is total reflection. What happens for $i>i_c$? Now for $i>i_c$, sinr>1, which implies that the angle becomes complex with purely imaginary cosine:

$$\cos(r) = j\sqrt{\left(\frac{\sin i}{\sin i_c}\right)^2 - 1} ,$$

where we have temporarily used j for $\sqrt{-1}$ instead of the almost universal i, in order not to confuse it with the angle of incidence. The meaning of these complex quantities becomes clear when we consider the propagation factor for the refracted wave:

$$e^{j\vec{k}'.\vec{x}} = e^{jk'(x\sin r + z\cos r)} = e^{-k'[(\sin i/\sin i_c)^2 - 1]^{1/2} z} e^{jk'(\sin i/\sin i_c)x}$$

This shows that for $i>i_c$, the refracted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface. This attenuation occurs within a very few wavelengths of the interface except for $i=i_c$.

Even though fields exist on the other side of the surface, there is no energy flow through the surface. Hence total internal reflection occurs for $i \geq i_c$. The fact that though there are fields but no energy flow can be verified by calculating the time-averaged normal component of the Poynting vector just inside the surface:

$$\vec{S}.\hat{n} = \frac{1}{2}\mathrm{Re}[\hat{n}.(\vec{E}'\times\vec{H}'^*)]$$

From equation (4)

$$\vec{B}' = \frac{c}{\omega}\vec{k}'\times\vec{E}' \Rightarrow \vec{H}' = \frac{c}{\mu'\omega}\vec{k}'\times\vec{E}',$$

so that

$$\vec{S}.\hat{n} = \frac{c^2}{2\omega\mu}\mathrm{Re}[(\hat{n}.\vec{k}')\left|\vec{E}_0'\right|^2]$$

But $(\hat{n}.\vec{k}')$ is purely imaginary, so that $\vec{S}.\hat{n} = 0$.

We will now apply the Fresnel's formulas to examine the process in detail. For electric field perpendicular to the plane of incidence (case – 1), equation (17) can be written as (remember, we now have $\mu = \mu'$)

$$\frac{E_0''}{E_0}=\frac{n\cos i-\sqrt{n'^2-n^2\sin^2 i}}{n\cos i+\sqrt{n'^2-n^2\sin^2 i}}=\frac{\cos i-j\sqrt{\sin^2 i-\frac{n'^2}{n^2}}}{\cos i+j\sqrt{\sin^2 i-\frac{n'^2}{n^2}}}$$

Since the numerator and denominator are complex conjugate of each other, it is evident that the reflected amplitude is equal in magnitude to the incident amplitude. The phase however is changed. Writing

$$\cos i-j\sqrt{\sin^2 i-\frac{n'^2}{n^2}}=e^{-j\varphi};\quad \varphi=\tan^{-1}\left(\frac{\sqrt{\sin^2 i-\frac{n'^2}{n^2}}}{\cos i}\right),$$

$$E_0''=E_0e^{-2j\varphi}$$

For the component whose electric vector is in the plane of incidence (case – 2), we have from equation (21)

$$\frac{E_0''}{E_0}=\frac{n'^2\cos i-n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+n\sqrt{n'^2-n^2\sin^2 i}}=\frac{\cos i-j(\frac{n}{n'})^2\sqrt{\sin^2 i-(\frac{n'}{n})^2}}{\cos i+j(\frac{n}{n'})^2\sqrt{\sin^2 i-(\frac{n'}{n})^2}}$$

Again, by the same reasoning, the magnitude of the reflected amplitude equals the magnitude of incident amplitude. The phase change in this case is

$$E_0''=E_0e^{-2j\psi};\ \psi=\tan^{-1}\left(\frac{\sqrt{\sin^2 i-(\frac{n'}{n})^2}}{(\frac{n'}{n})^2\cos i}\right)$$

Thus, though the magnitude of the reflected wave remains unchanged in both the cases, the phases change and that too by unequal amounts, and as a result, plane polarized radiation on total internal reflection is returned back with the same intensity but as elliptically polarized radiation. The relative change of phase of the two components, 2(φ – ψ) depends both on the angle of incidence and the ratio, (n'/n). Now on using the above expressions for φ and ψ

$$\tan(\varphi-\psi)=\frac{\tan\varphi-\tan\psi}{1+\tan\phi\tan\psi}=-\frac{\cos i\sqrt{\sin^2 i-(\frac{n'}{n})^2}}{\sin^2 i}$$

The relative phase difference $2(\phi-\psi)$ is thus $-2\tan^{-1}\left(\frac{\cos i\sqrt{\sin^2 i-(\frac{n'}{n})^2}}{\sin^2 i}\right)$.

As we have seen, although the transmission coefficient is zero for total reflection, the fields do not vanish in the second medium; they fall off exponentially and in fact can be detected.

Summary

1. In this module we describe to the student the phenomenon of reflection and refraction of an electromagnetic wave when it falls on the interface between two media.
2. We then obtain the kinematic relations between the angles of incidence, reflection and refraction; these relations are independent of the state of polarization of the incident wave.

3. We next obtain the dynamic properties, the intensities of the refracted and refracted waves. These intensities depend on the nature of polarization of the incident beam.
4. It is explained as to how an unpolarized incident electromagnetic wave gets polarized on reflection and refraction.
5. Finally the phenomenon of total internal reflection and the conditions required for it are discussed.